

Meson-baryon interaction in the meson exchange picture

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Nucl. Phys. A **829**, 170 (2009), Phys. Lett. B **681**, 26 (2009)

The Jülich model of pion-nucleon interaction

Motivation

- ▶ Hadronic reactions provide insight to QCD in the non-perturbative region.
 - ▶ Intense experimental effort at JLab (Clas), ELSA, MAMI, ...
 - ▶ Theoretical data analysis required (e.g. EBAC/JLab).
-
- ▶ Chiral Lagrangian of Wess and Zumino.
 - ▶ Channels πN , ηN ; effective $\pi\pi N$ channels σN , ρN , $\pi\Delta$.
 - ▶ Baryonic resonances up to $J = 3/2$ with derivative couplings as required by chiral symmetry.
 - ▶ New: **KY** channels, $J = 5/2, 7/2$ resonances, **chiral unitary** σ meson.

Other non-K-matrix meson exchange models: DMT, Surya, Gross, PRC 53 (1996); **EBAC** Juliá Díaz et al., PRC 76 (2007); Suzuki, Sato, Lee, PRC 79 (2009); Kamano et al., PRC 79 (2009).

Scattering equation in the *JLS* basis

$$\langle L'S'k' | \textcolor{red}{T}_{\mu\nu}^{IJ} | LSk \rangle = \langle L'S'k' | \textcolor{red}{V}_{\mu\nu}^{IJ} | LSk \rangle \\ + \sum_{\gamma, L''S''} \int_0^{\infty} k''^2 dk'' \langle L'S'k' | \textcolor{red}{V}_{\mu\gamma}^{IJ} | L''S''k'' \rangle \frac{1}{Z - E_{\gamma}(k'') + i\epsilon} \langle L''S''k'' | \textcolor{red}{T}_{\gamma\nu}^{IJ} | LSk \rangle$$

T: Amplitude

V: Pseudopotential

G: Propagator

J: total angular momentum

L: orbital angular momentum

S: total Spin of MB system

I: isospin

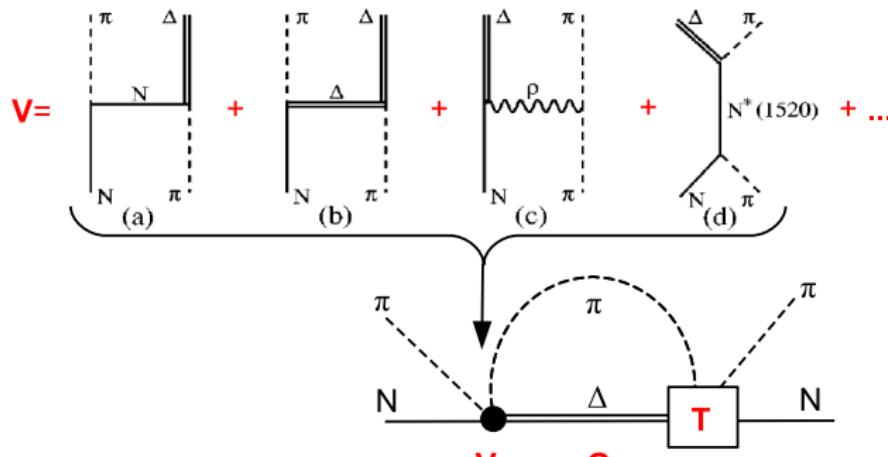
$k(k', k'')$: incoming(outgoing, intermediate) momentum, on- or off-shell

$\mu(\nu, \gamma)$: incoming(outgoing, intermediate) channel $[\pi N, \eta N, \pi\Delta, \rho N, \sigma N]$

Scattering equation in the *JLS* basis

$$\langle L'S'k' | \mathbf{T}_{\mu\nu}^{IJ} | LSk \rangle = \langle L'S'k' | \mathbf{V}_{\mu\nu}^{IJ} | LSk \rangle$$

$$+ \sum_{\gamma, L''S''} \int_0^{\infty} k''^2 dk'' \langle L'S'k' | \mathbf{V}_{\mu\gamma}^{IJ} | L''S''k'' \rangle \frac{1}{Z - E_{\gamma}(k'') + i\epsilon} \langle L''S''k'' | \mathbf{T}_{\gamma\nu}^{IJ} | LSk \rangle$$



Scattering equation in the *JLS* basis

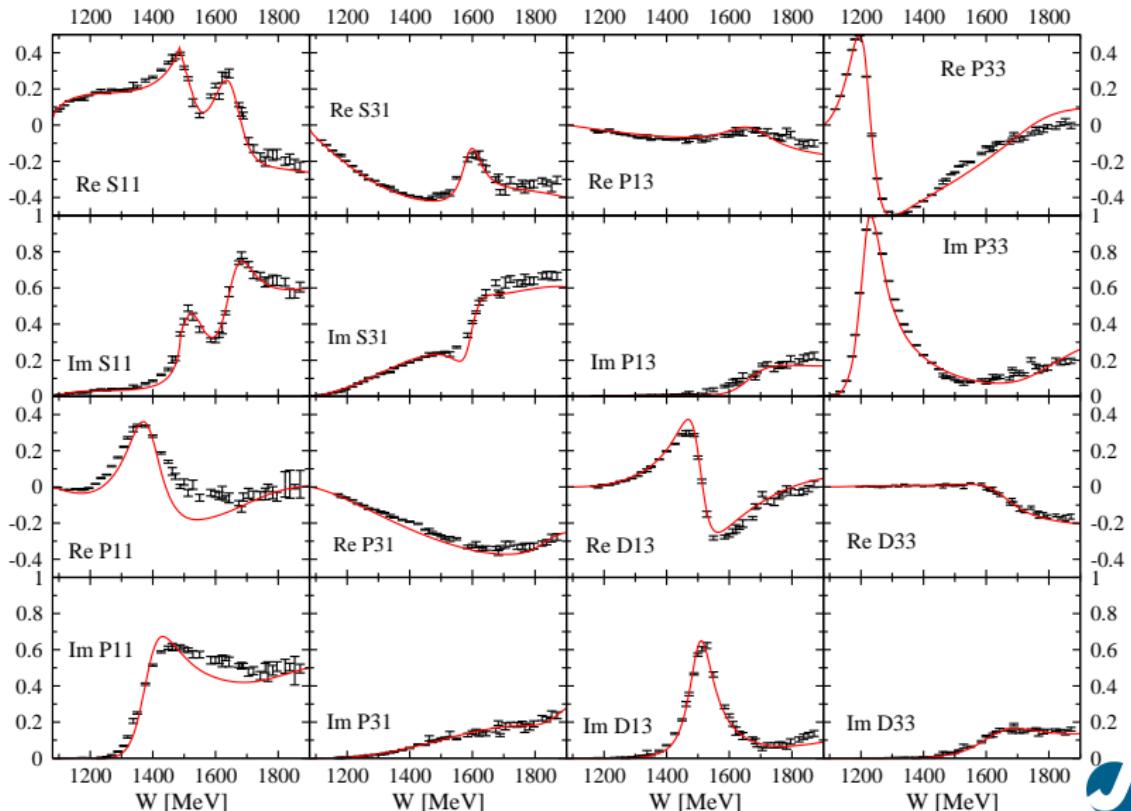
$$\langle L'S'k' | \textcolor{red}{T}_{\mu\nu}^{IJ} | LSk \rangle = \langle L'S'k' | \textcolor{red}{V}_{\mu\nu}^{IJ} | LSk \rangle \\ + \sum_{\gamma, L''S''} \int_0^{\infty} k''^2 dk'' \langle L'S'k' | \textcolor{red}{V}_{\mu\gamma}^{IJ} | L''S''k'' \rangle \frac{1}{Z - E_{\gamma}(k'') + i\epsilon} \langle L''S''k'' | \textcolor{red}{T}_{\gamma\nu}^{IJ} | LSk \rangle$$

Features

- ▶ Meson exchange provides the correct and relevant degrees of freedom in the second and third resonance region (no contact terms).
- ▶ No on-shell factorization. Full analyticity (dispersive parts).
- ▶ Meson exchange puts strong constraints as all partial waves are linked.
- ▶ Dynamical generation of resonances is possible, but not easy (strong constraints!)
- ▶ Topic of missing resonances: no need to model background with resonances.

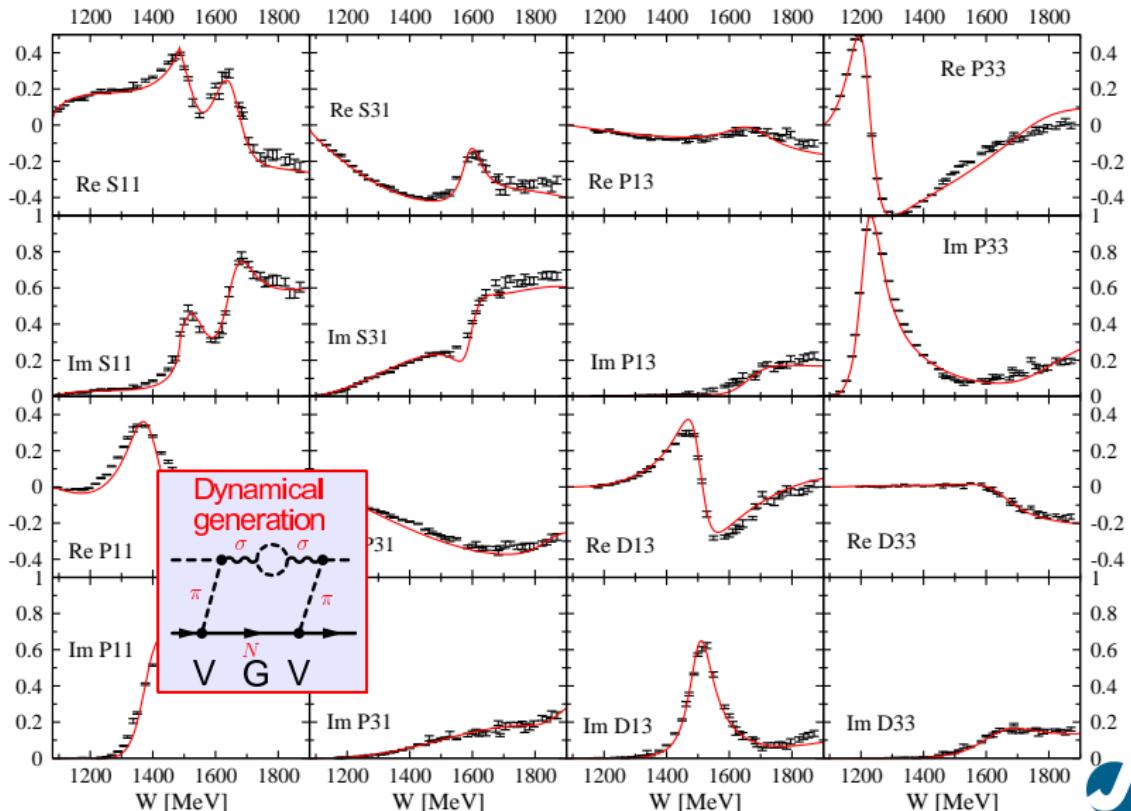
Partial waves in $\pi N \rightarrow \pi N$ (Solution 2002)

"Data": GWU/SAID analysis, single energy solution



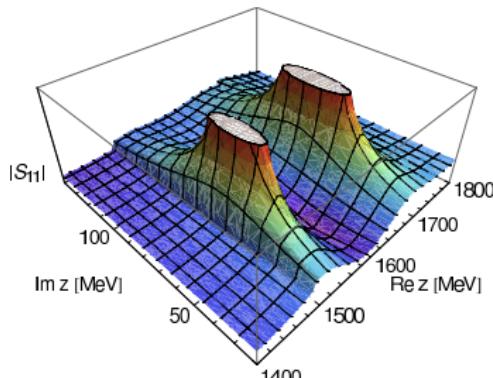
Partial waves in $\pi N \rightarrow \pi N$ (Solution 2002)

"Data": GWU/SAID analysis, single energy solution



Poles and residues

on the unphysical sheets, given by the analytic continuation.



	$\text{Re } z_0$ [MeV]	$-2 \text{Im } z_0$ [MeV]	$ R $ [MeV]	θ [deg] $[^\circ]$
$N^*(1440) P_{11}$	1387	147	48	-64
ARN	1359	162	38	-98
HOE	1385	164	40	
CUT	1375 ± 30	180 ± 40	52 ± 5	-100 ± 35
$N^*(1520) D_{13}$	1505	95	32	-18
ARN	1515	113	38	-5
HOE	1510	120	32	-8
CUT	1510 ± 5	114 ± 10	35 ± 2	-12 ± 5

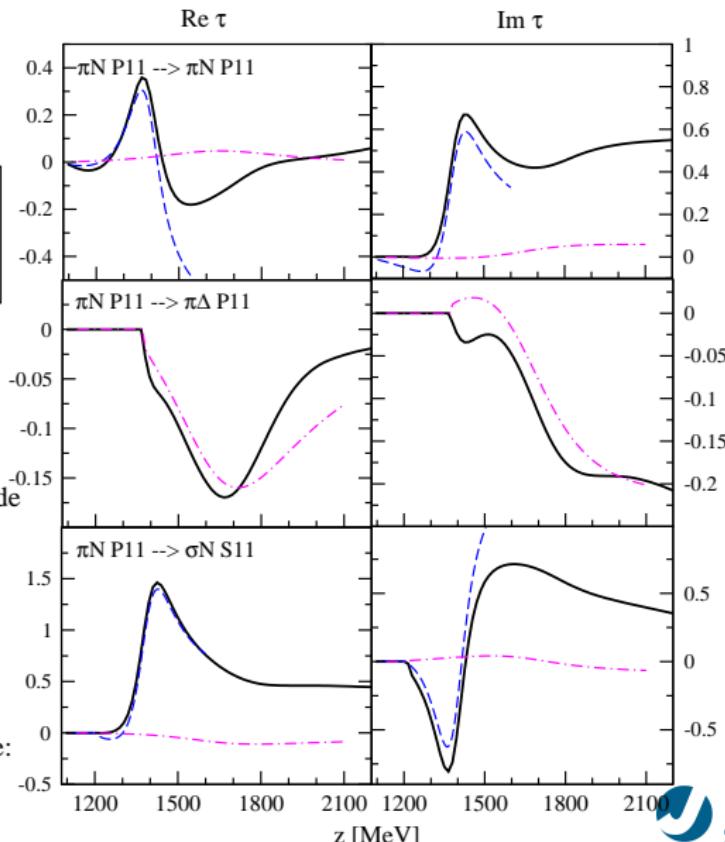
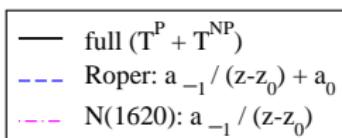
	$\text{Re } z_0$ [MeV]	$-2 \text{Im } z_0$ [MeV]	$ R $ [MeV]	θ [deg] $[^\circ]$
$N^*(1535) S_{11}$	1519	129	31	-3
ARN	1502	95	16	-16
HOE	1487			
CUT	1510 ± 50	260 ± 80	120 ± 40	+15 ± 45
$N^*(1650) S_{11}$	1669	136	54	-44
ARN	1648	80	14	-69
HOE	1670	163	39	-37
CUT	1640 ± 20	150 ± 30	60 ± 10	-75 ± 25
$N^*(1720) P_{13}$	1663	212	14	-82
ARN	1666	355	25	-94
HOE	1686	187	15	
CUT	1680 ± 30	120 ± 40	8 ± 12	-160 ± 30
$\Delta(1232) P_{33}$	1218	90	47	-37
ARN	1211	99	52	-47
HOE	1209	100	50	-48
CUT	1210 ± 1	100 ± 2	53 ± 2	-47 ± 1
$\Delta^*(1620) S_{31}$	1593	72	12	-108
ARN	1595	135	15	-92
HOE	1608	116	19	-95
CUT	1600 ± 15	120 ± 20	15 ± 2	-110 ± 20
$\Delta^*(1700) D_{33}$	1637	236	16	-38
ARN	1632	253	18	-40
HOE	1651	159	10	
CUT	1675 ± 25	220 ± 40	13 ± 3	-20 ± 25
$\Delta^*(1910) P_{31}$	1840	221	12	-153
ARN	1771	479	45	+172
HOE	1874	283	38	
CUT	1880 ± 30	200 ± 40	20 ± 4	-90 ± 30

[ARN]: Arndt et al., PRC 74 (2006), [HOE]: Höhler, πN Newslet. 9 (1993), [CUT]: Cutkowski et al., PRD 20 (1979).

Analytic continuation, residues to ηN , σN , ρN , $\pi \Delta$. Zeros. Branching ratios to πN , ηN .

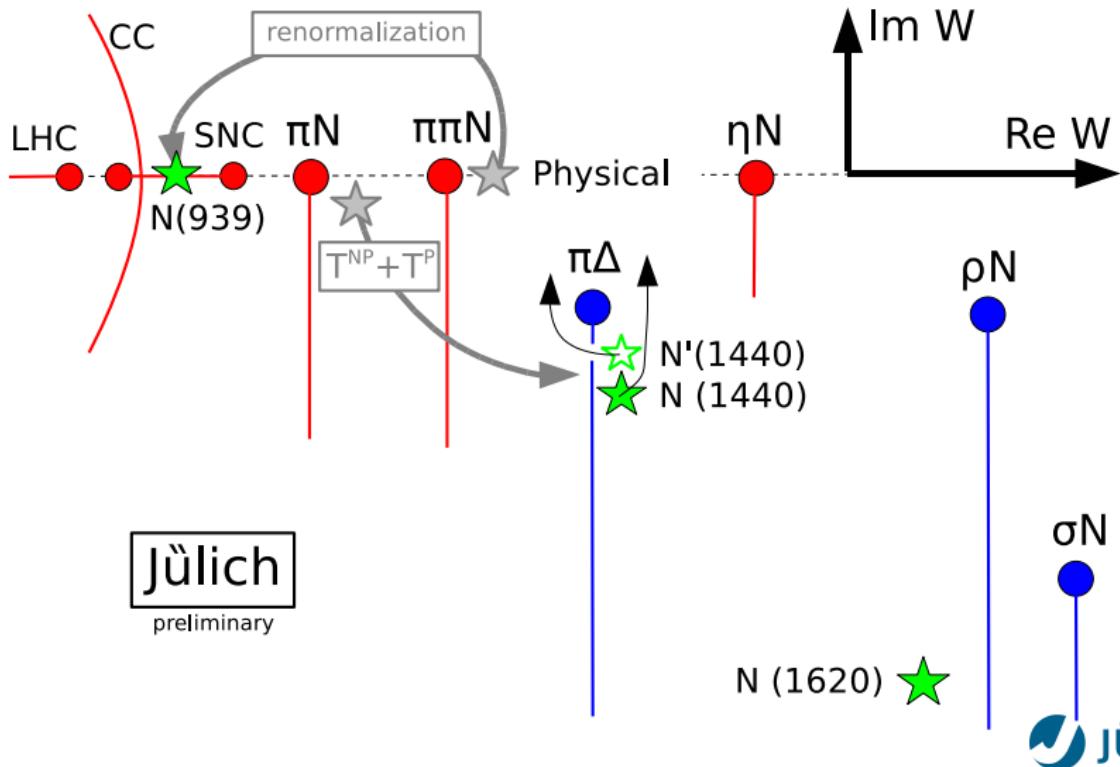
An additional state in P11

* New pole in P11 found
at $z=1620 + 297 i$ MeV.



- * Very weak branching to πN .
- * Very large branching to $\pi \Delta$.
- * Resonance transition amplitude
 $\pi N \rightarrow \pi \Delta$:
 Manley ($\pi N \rightarrow \pi\pi N$): -0.21
 here: -0.25
- * Roper: generated from σN .
- * N(1620): generated from $\pi \Delta$.
- * Dynamics of the nucleon pole:
 Pole repulsion $N \leftrightarrow$ Roper

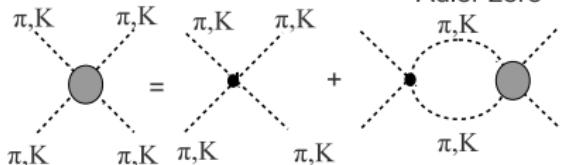
The analytic structure of the P11 partial wave



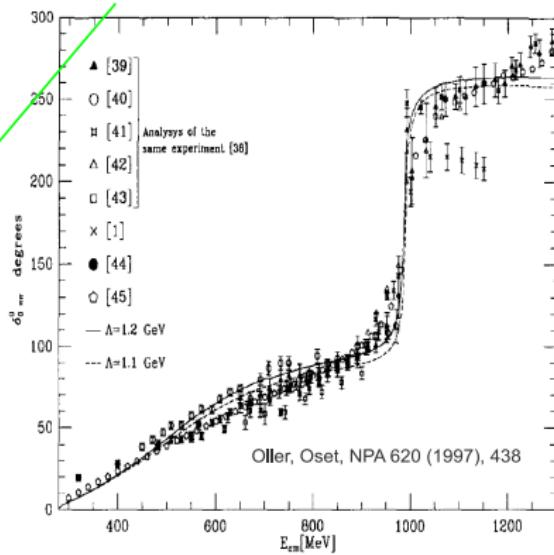
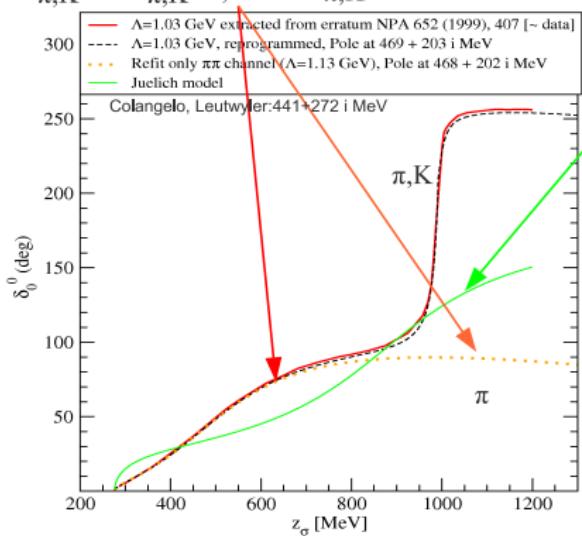
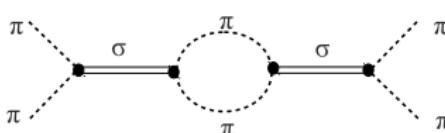
Chiral unitary approach to $\pi\pi$ scattering

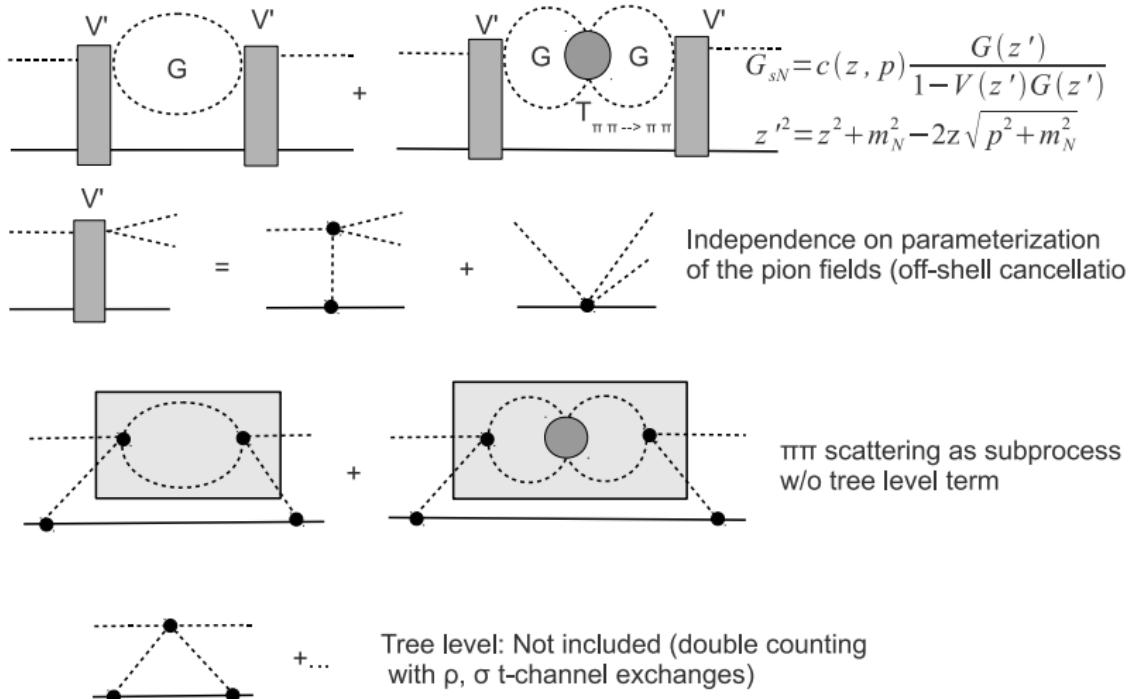
$$T_{\pi\pi} = V\chi + V\chi G T_{\pi\pi}; V\chi = \frac{(1/2m_\pi^2 - s)}{f^2}$$

Adler zero



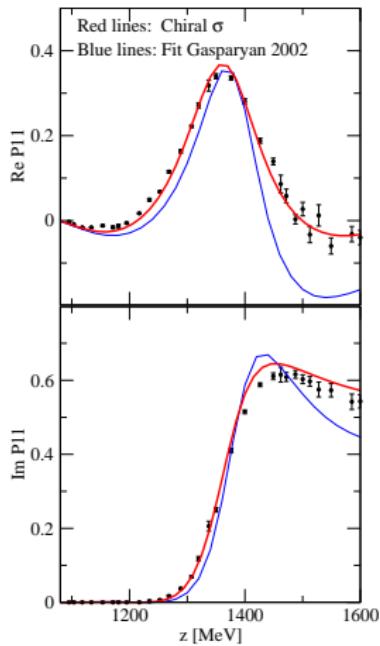
$$T_{\pi\pi} = (V_{\sigma\pi\pi})^2 / (z - M - \Sigma_{\pi\pi})$$



Implementation of the chiral σ 

Result for the Roper resonance

Readjustment of cut-offs and $g_{\sigma\sigma\sigma}$ coupling



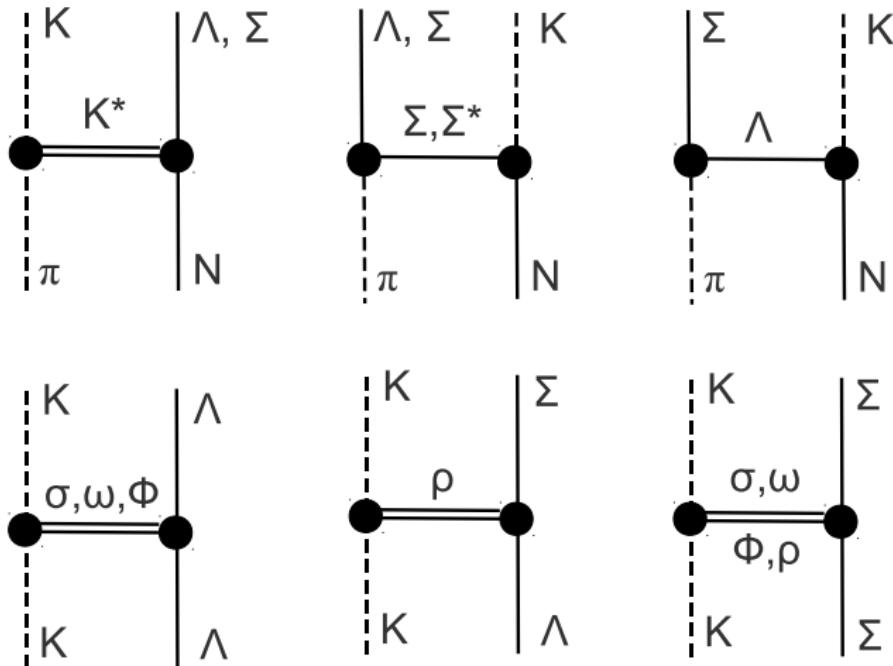
- ▶ Analytic structure of the amplitude exactly the same as before (3 branchpoints from σN)
- ▶ Dynamical generation of Roper and second P11 does not depend on details of the model
- ▶ Chiral σ provides better description.

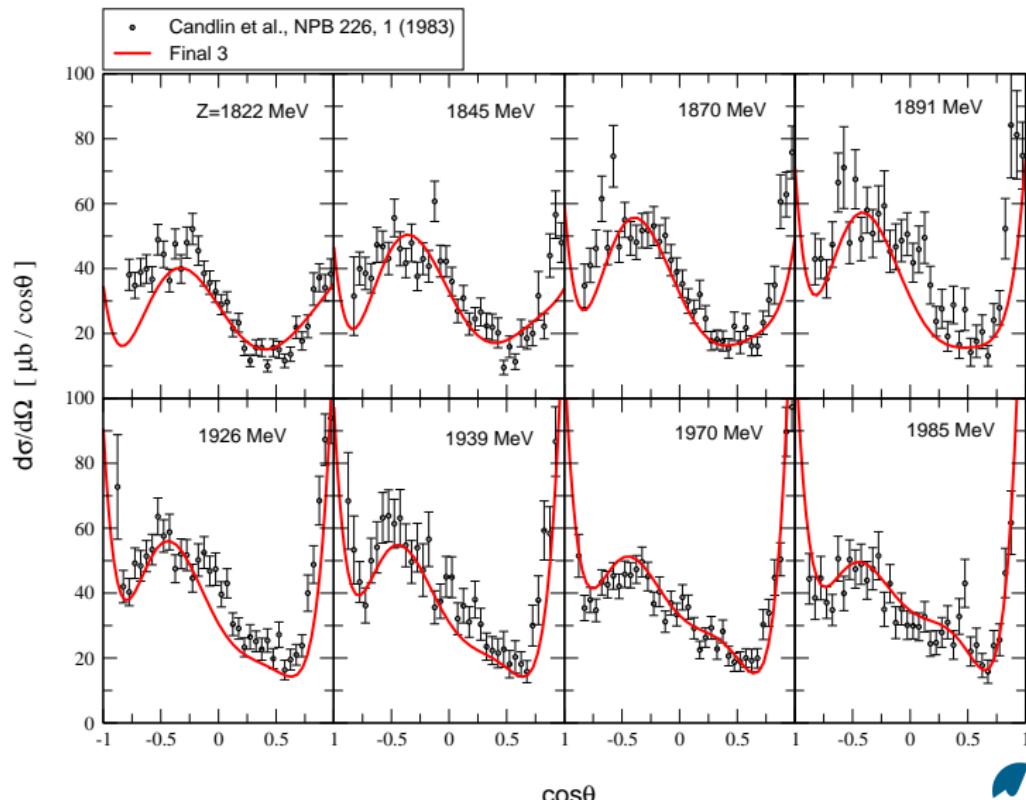
The reaction $\pi^+ p \rightarrow K^+ \Sigma^+$

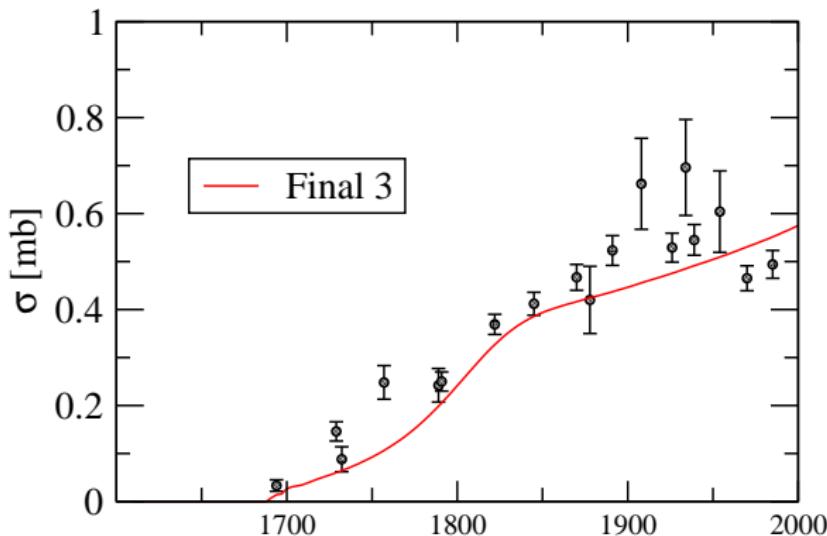
- ▶ Pure isospin 3/2; relatively few Δ resonances.
- ▶ → Strong constraints on the amplitude of the Jülich model!
- ▶ Good, simple data situation (Candlin 1983-1988).
- ▶ Simultaneous fit to $\pi N \rightarrow \pi N$ partial waves plus $\pi^+ p \rightarrow K^+ \Sigma^+$ observables.
- ▶ Requires extended fitting efforts: Parallelization of the code in energies, implementation of Minuit done.
- ▶ Code runs on Juropa@Jülich.
- ▶ First tasks:
 - ▶ Inclusion of KY in $t-$ and $u-$ channel exchange processes.
 - ▶ Inclusion of Spin 5/2 and Spin 7/2 resonances.

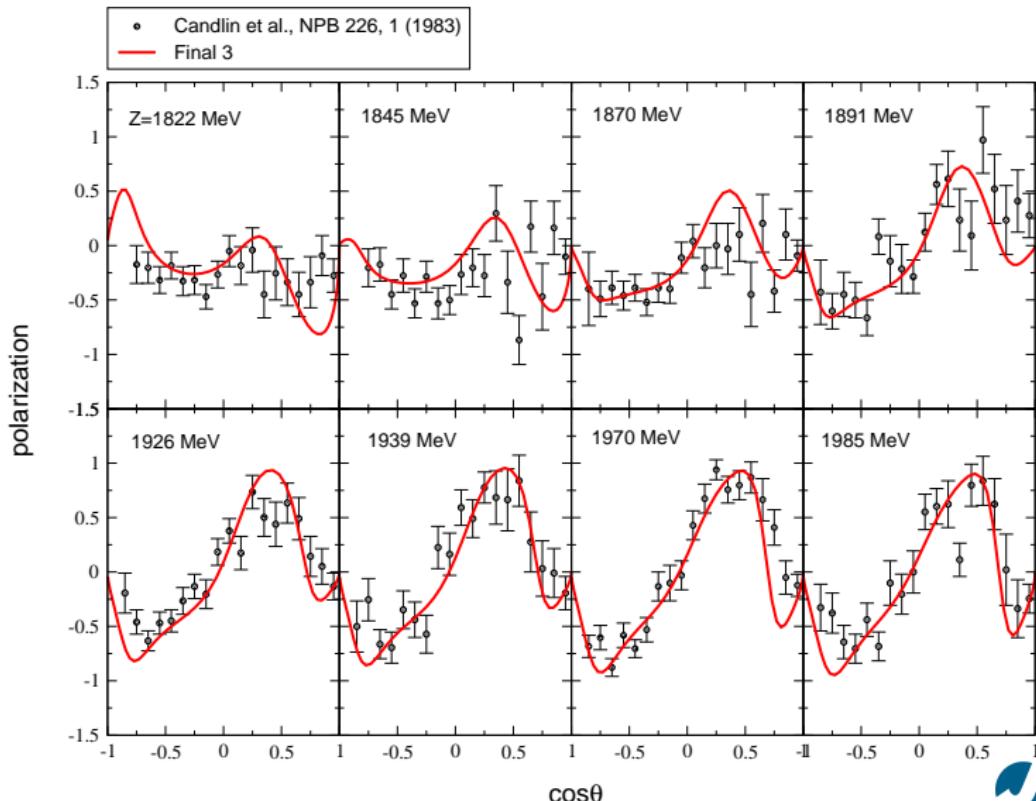
Inclusion of the KY channels

Inclusion via $SU(3)$ symmetry [no additional freedom]



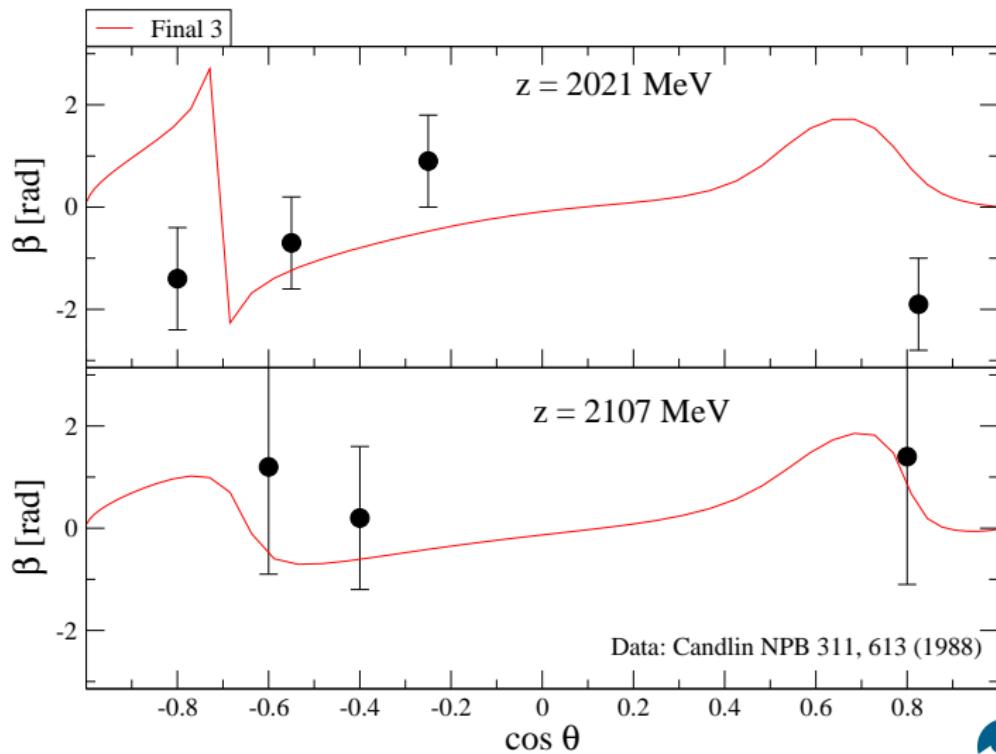
Differential cross section of $\pi^+ p \rightarrow K^+ \Sigma^+$ 

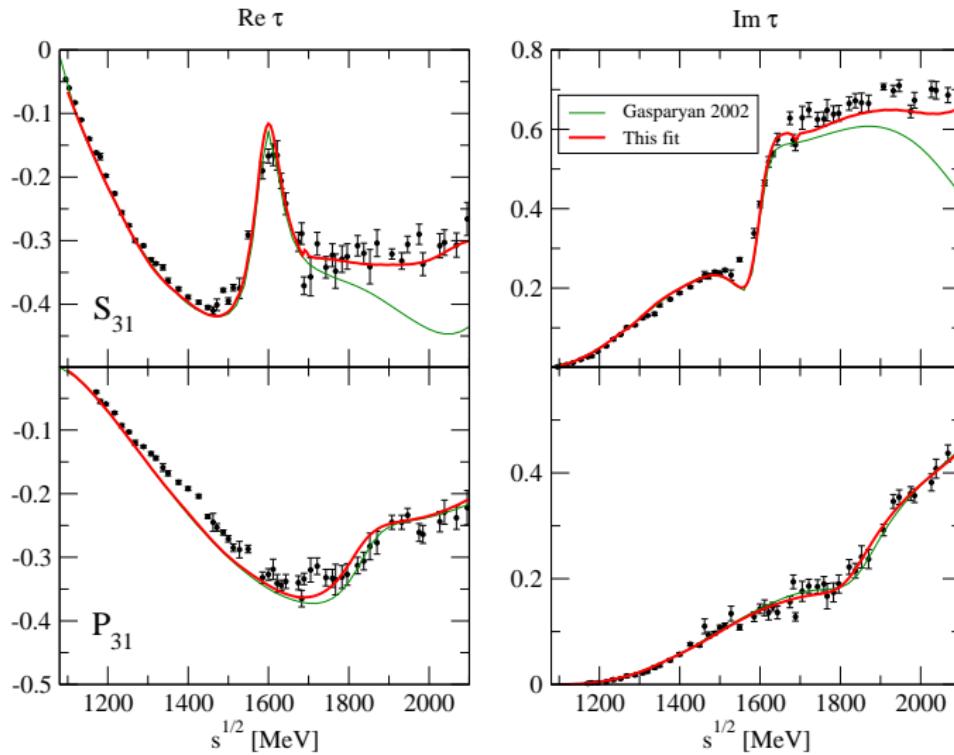
Total cross section of $\pi^+ p \rightarrow K^+ \Sigma^+$ 

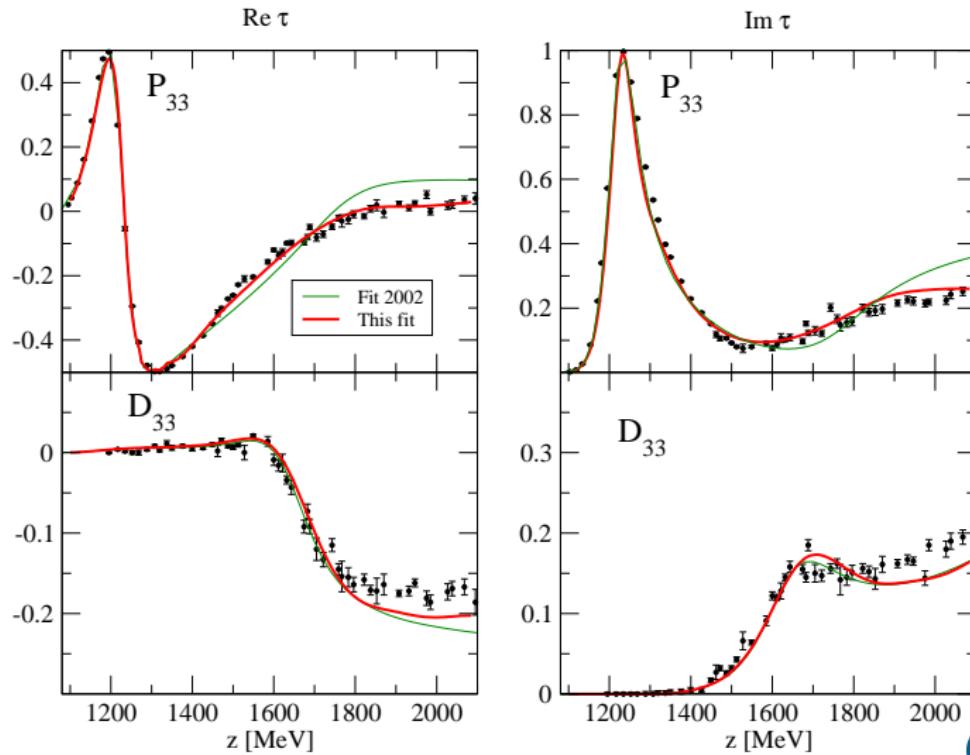
Polarization of $\pi^+ p \rightarrow K^+ \Sigma^+$ 

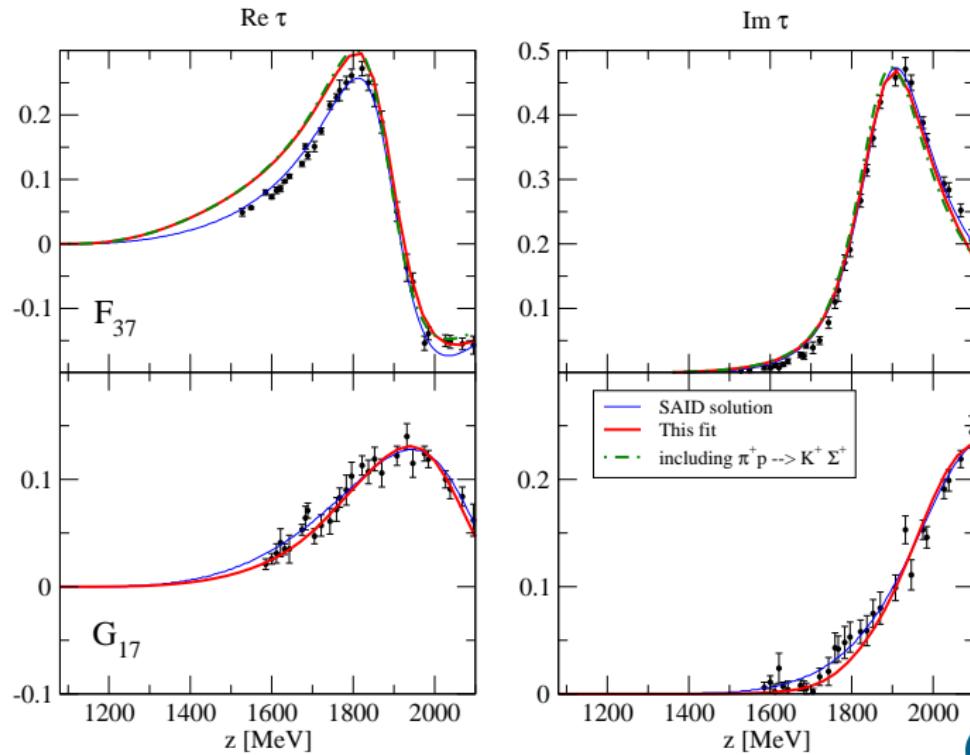
Spin rotation parameter β of $\pi^+ p \rightarrow K^+ \Sigma^+$

Definitions Observables



$\pi N \rightarrow \pi N$ phase shiftsSimultaneous fit to $\pi^+ p \rightarrow K^+ \Sigma^+$ 

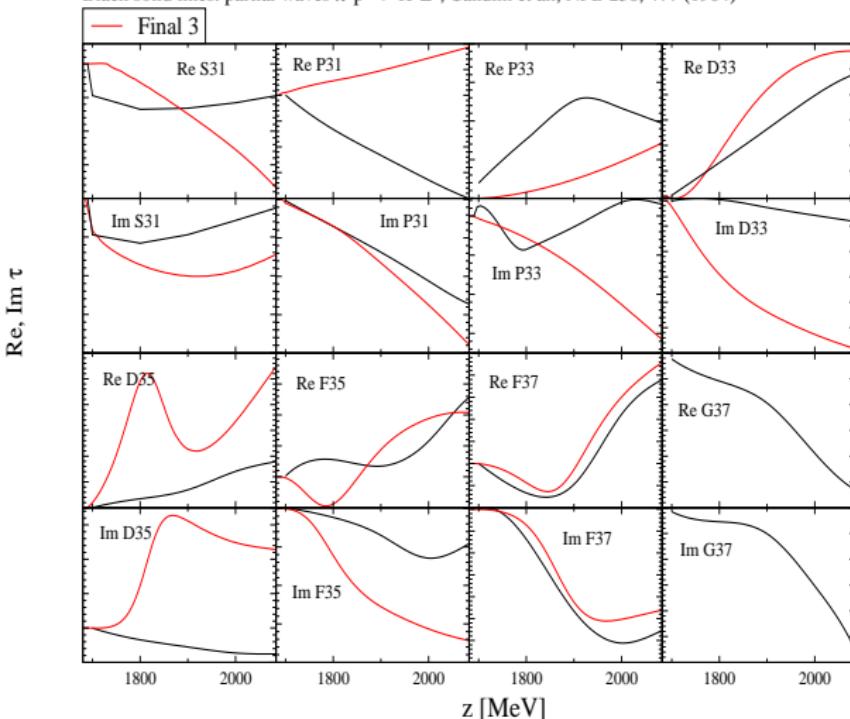
$\pi N \rightarrow \pi N$ phase shiftsSimultaneous fit to $\pi^+ p \rightarrow K^+ \Sigma^+$ 

$\pi N \rightarrow \pi N$ phase shifts (higher partial waves; selection)Simultaneous fit to $\pi^+ p \rightarrow K^+ \Sigma^+$ 

The $\pi^+ p \rightarrow K^+\Sigma^+$ partial wave amplitudes of Candlin 1984

(Comparable quality of the fit)

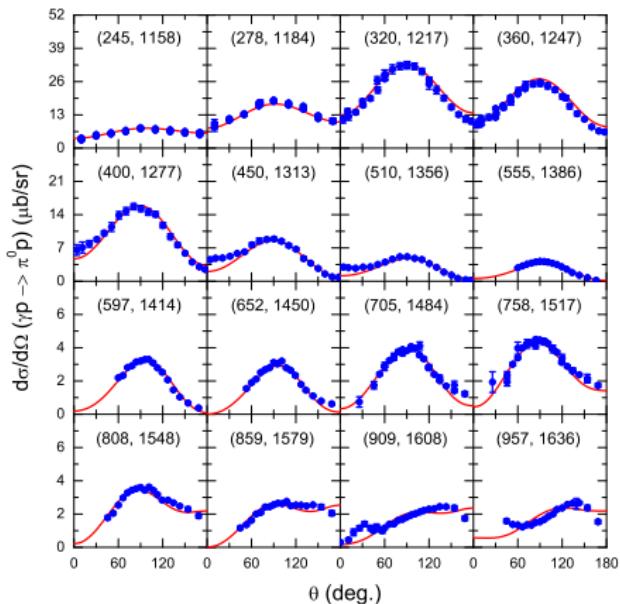
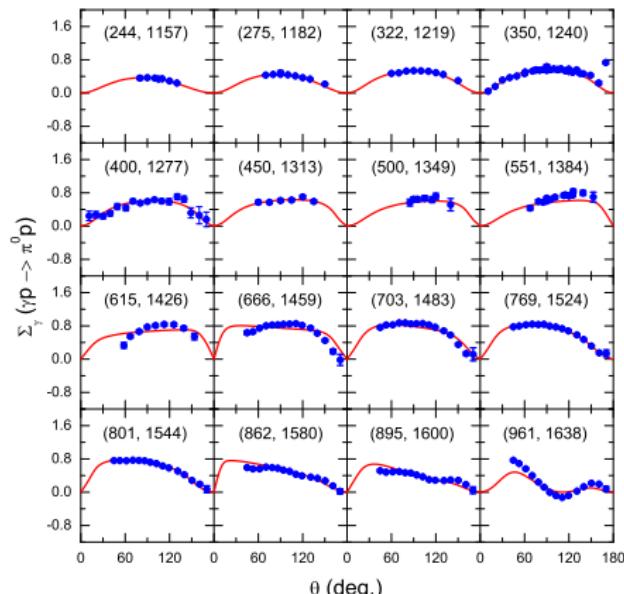
Black solid lines: partial waves $\pi^+ p \rightarrow K^+\Sigma^+$, Candlin et al., NPB 238, 477 (1984)



- ▶ Very different partial wave content!
- ▶ Additional constraints from $\pi N \rightarrow \pi N$ indeed necessary.

New results on photoproduction (Talk of Fei Huang, S. Krewald)

Full gauge invariance respected!

Differential cross section for $\gamma p \rightarrow \pi^0 p$ Photon spin asymmetry for $\gamma p \rightarrow \pi^0 p$

Conclusions

- ▶ Lagrangian based, field theoretical description of meson-baryon interaction.
- ▶ (Some) chiral constraints are fulfilled (σ meson). Unitarity and full analyticity are ensured.
- ▶ Analytic continuation: precise, model independent determination of resonance parameters (poles).
- ▶ Roper dynamically generated; hints for additionally generated structures.
Examples: $P11(1710)$ ($\pi\Delta$), $D13(1700)$ (ρN from hidden gauge).
- ▶ Meson and baryon exchange are the relevant degrees of freedom in the 2nd and 3rd resonance region. Exchange provides heavy constraints, because all partial waves are linked. With this background, only a few bare states are needed (so far).
- ▶ Coupled channel formalism links different reactions in one combined description:
 - ▶ $\pi N \rightarrow \pi N$, $\pi^+ p \rightarrow K^+ \Sigma^+$, ..., $\gamma N \rightarrow \pi N$, ...

Analytic continuation via Contour deformation

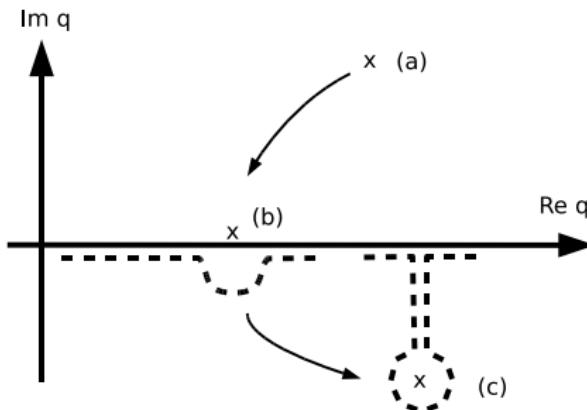
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...enables access to all Riemann sheets

$$\Pi_\sigma(z) = \int_0^\infty q^2 dq \frac{(v^{\sigma\pi\pi}(q))^2}{z - E_1 - E_2 + i\epsilon}$$

$$z - E_1 - E_2 = 0 \Leftrightarrow q = q_{c.m.}$$

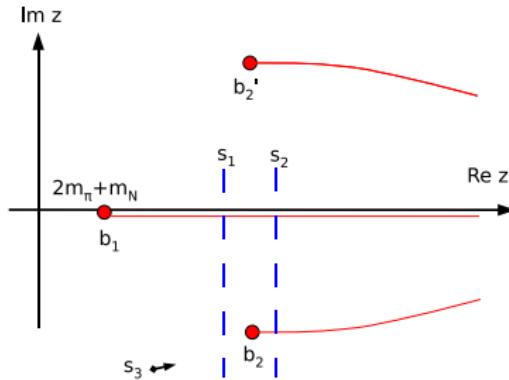
$$q_{c.m.} = \frac{1}{2z} \sqrt{[z^2 - (m_1 - m_2)^2][z^2 - (m_1 + m_2)^2]}$$



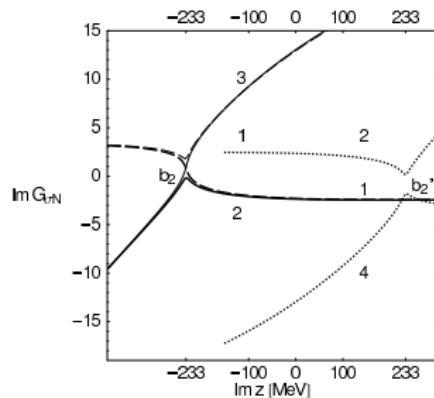
- ▶ Plot $q_{c.m.}(z)$ in the q plane of integration (X: Pole positions).
- ▶ case (a), $\text{Im } z > 0$: straight integration from $q = 0$ to $q = \infty$.
- ▶ case (b), $\text{Im } z = 0$: Pole is on real q axis.
- ▶ case (c), $\text{Im } z < 0$: Deformation gives analytic continuation.
- ▶ Special case: Pole at $q = 0$ \Leftrightarrow branch point at $z = m_1 + m_2$ (= threshold).

Effective $\pi\pi N$ channels: Analytic structure

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Three branch points and four sheets for each of the σN , ρN , and $\pi\Delta$ propagators.

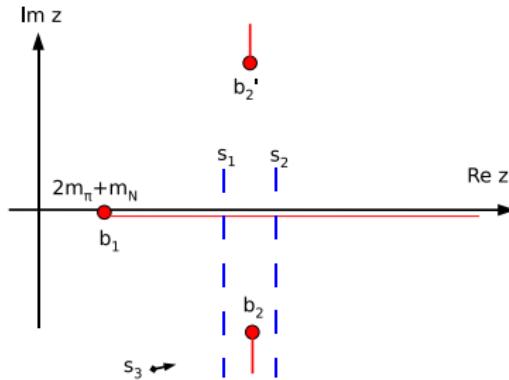


- ▶ The cut along $\text{Im } z = 0$ is induced by the cut of the self energy of the unstable particle.
- ▶ The poles of the unstable particle (σ) induce branch points in the σN propagator at

$$z_{b_2} = m_N + z_0, z_{b_2'} = m_N + z_0^*$$

Effective $\pi\pi N$ channels: Analytic structure

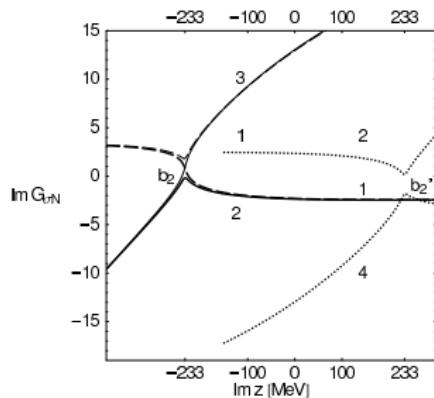
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Three branch points and four sheets for each of the σN , ρN , and $\pi \Delta$ propagators.



Couplings “ $g = \sqrt{a_{-1}}$ ” to other channels

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	$N\pi$	$N\rho^{(1)} (S = 1/2)$	$N\rho^{(2)} (S = 3/2)$	$N\rho^{(3)} (S = 3/2)$
$N^*(1535) S_{11}$	$S_{11} \quad 8.1 + 0.5i$	$S_{11} \quad 2.2 - 5.4i$	—	$D_{11} \quad 0.5 - 1.3i$
$N^*(1650) S_{11}$	$S_{11} \quad 8.6 - 2.8i$	$S_{11} \quad 0.9 - 9.1i$	—	$D_{11} \quad 0.3 - 2.0i$
$N^*(1440) P_{11}$	$P_{11} \quad 11.2 - 5.0i$	$P_{11} \quad -1.3 + 3.2i$	$P_{11} \quad 3.6 - 2.6i$	—
$\Delta^*(1620) S_{31}$	$S_{31} \quad 2.9 - 3.7i$	$S_{31} \quad 0.0 - 0.0i$	—	$D_{31} \quad 0.0 + 0.5i$
$\Delta^*(1910) P_{31}$	$P_{31} \quad 1.2 - 3.5i$	$P_{31} \quad 0.2 - 0.4i$	$P_{31} \quad -0.2 - 0.4i$	—
$N^*(1720) P_{13}$	$P_{13} \quad 3.7 - 2.6i$	$P_{13} \quad 0.1 + 0.8i$	$P_{13} \quad -1.1 + 0.1i$	$F_{13} \quad 0.1 + 0.4i$
$N^*(1520) D_{13}$	$D_{13} \quad 8.4 - 0.8i$	$D_{13} \quad -0.6 + 0.7i$	$D_{13} \quad 0.9 - 2.0i$	$S_{13} \quad -2.5 - 22.8i$
$\Delta(1232) P_{33}$	$P_{33} \quad 17.9 - 3.2i$	$P_{33} \quad -1.3 - 0.8i$	$P_{33} \quad -0.9 - 3.0i$	$F_{33} \quad 0.0 - 0.0i$
$\Delta^*(1700) D_{33}$	$D_{33} \quad 4.9 - 1.0i$	$D_{33} \quad -0.2 + 0.9i$	$D_{33} \quad -0.4 - 0.4i$	$S_{33} \quad -0.1 - 0.9i$

	$N\eta$	$\Delta\pi^{(1)}$	$\Delta\pi^{(2)}$	$N\sigma$
$N^*(1535) S_{11}$	$S_{11} \quad 11.9 - 2.3i$	—	$D_{11} \quad -5.9 + 4.8i$	$P_{11} \quad -1.4 - 1.2i$
$N^*(1650) S_{11}$	$S_{11} \quad -3.0 + 0.5i$	—	$D_{11} \quad 4.3 + 0.4i$	$P_{11} \quad -2.1 - 1.0i$
$N^*(1440) P_{11}$	$P_{11} \quad -0.1 + 0.0i$	$P_{11} \quad -4.6 - 1.7i$	—	$S_{11} \quad -8.3 - 27.7i$
$\Delta^*(1620) S_{31}$	—	—	$D_{31} \quad 11.1 - 4.0i$	—
$\Delta^*(1910) P_{31}$	—	$P_{31} \quad 15.0 - 0.3i$	—	—
$N^*(1720) P_{13}$	$P_{13} \quad -7.7 + 5.5i$	$P_{13} \quad -14.1 + 3.0i$	$F_{13} \quad 0.0 - 0.3i$	$D_{13} \quad -0.8 + 0.4i$
$N^*(1520) D_{13}$	$D_{13} \quad 0.16 - 0.60i$	$D_{13} \quad 0.0 + 0.4i$	$S_{13} \quad -12.9 - 0.7i$	$P_{13} \quad -0.6 - 0.6i$
$\Delta(1232) P_{33}$	—	$P_{33} \quad -(4 \text{ to } 5) + i(0 \text{ to } 0.5)$	$F_{33} \quad \sim 0$	—
$\Delta^*(1700) D_{33}$	—	$D_{33} \quad -0.7 - 0.3i$	$S_{33} \quad -19.7 + 4.5i$	—

Resonance couplings $g_i [10^{-3} \text{ MeV}^{-1/2}]$ to the coupled channels i . Also, the LJS type of each coupling is indicated. For the ρN channels, the total spin S is also indicated.

Zeros and branching ratio to πN , ηN

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first sheet	second sheet	[FA02]
P_{11}	$1235 - 0 i$	$1587 - 45 i$
D_{33}	$1396 - 78 i$	$1585 - 17 i$
		$1848 - 83 i$
		$1607 - 38 i$
		$1702 - 64 i$
		$1702 - 64 i$

Position of **zeros** of the full amplitude T in [MeV]. [FA02]: Arndt et al., PRC 69 (2004).

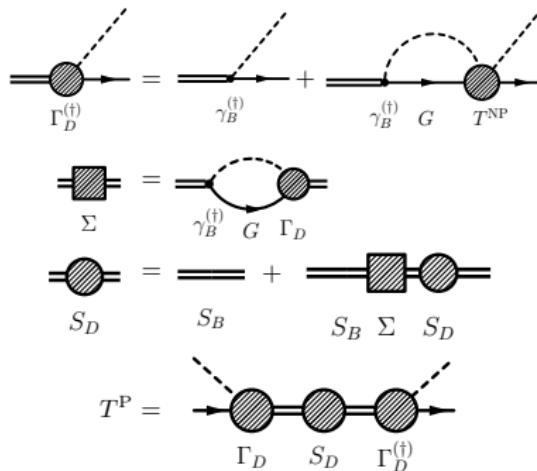
	$\Gamma_{\pi N}/\Gamma_{\text{Tot}} [\%]$	$\Gamma_{\eta N}/\Gamma_{\text{Tot}} [\%]$
$N^*(1535) S_{11}$	48 [33 to 55]	38 [45 to 60]
$N^*(1650) S_{11}$	79 [60 to 95]	6 [3 to 10]
$N^*(1440) P_{11}$	64 [55 to 75]	0 [0 ± 1]
$\Delta^*(1620) S_{31}$	34 [20 to 30]	—
$\Delta^*(1910) P_{31}$	11 [15 to 30]	—
$N^*(1720) P_{13}$	13 [10 to 20]	38 [4 ± 1]
$N^*(1520) D_{13}$	67 [55 to 65]	0.10 [0.23 ± 0.04]
$\Delta(1232) P_{33}$	100 [100]	—
$\Delta^*(1700) D_{33}$	13 [10 to 20]	—

Branching ratios into πN and ηN . The values in brackets are from the PDG,
[Amsler et al., PLB 667 (2008)].

Couplings and dressed vertices

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Residue a_{-1} vs. dressed vertex Γ vs. bare vertex γ .



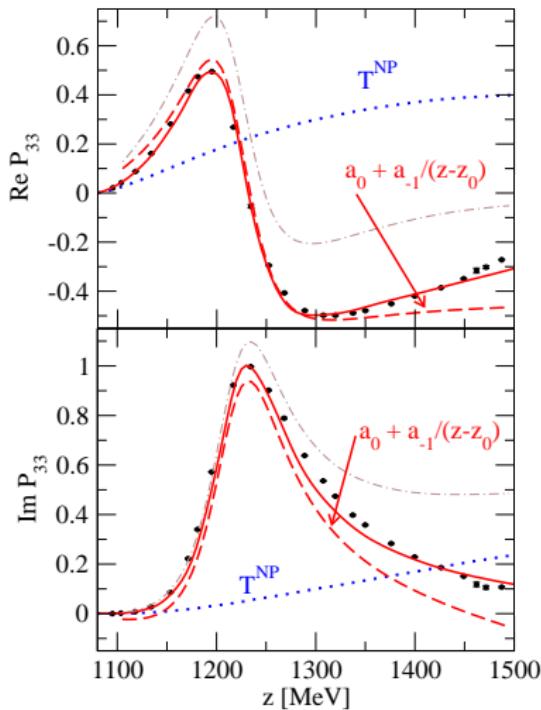
$$\begin{aligned} a_{-1} &= \frac{\Gamma_d \Gamma_d^{(\dagger)}}{1 - \frac{\partial}{\partial Z} \Sigma} \\ g &= \sqrt{a_{-1}} \\ r &= |(\Gamma_D - \gamma_B)/\Gamma_D|, \\ r' &= |1 - \sqrt{1 - \Sigma'}|, \end{aligned}$$

- Dressed Γ depends on T^{NP} .
- $\sqrt{a_{-1}} \neq \Gamma \neq \gamma$

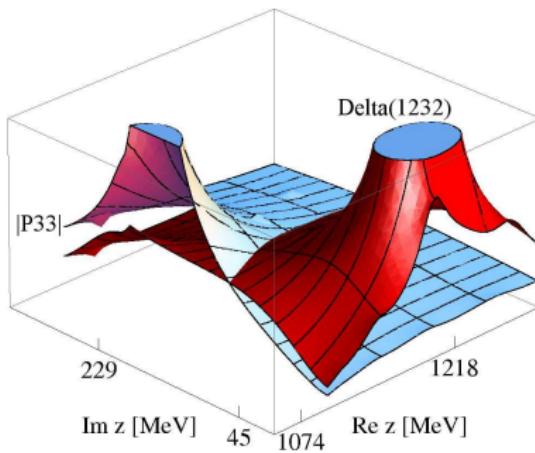
	γ^C	Γ^C	r [%]	r' [%]
$N^*(1520) D_{13}$	$6.4 - 0.6i$	$13.2 + 1.2i$	53	61
$N^*(1720) P_{13}$	$-0.1 + 5.4i$	$0.9 + 4.8i$	24	45
$\Delta(1232) P_{33}$	$1.3 + 13.0i$	$-2.8 + 22.2i$	45	40
$\Delta^*(1620) S_{31}$	$0.1 + 14.3i$	$5.0 + 5.7i$	130	66
$\Delta^*(1700) D_{33}$	$5.4 - 0.8i$	$6.7 + 1.0i$	33	54
$\Delta^*(1910) P_{31}$	$9.4 + 0.3i$	$1.9 - 3.2i$	222	22

Pole repulsion in P_{33}

◀ back



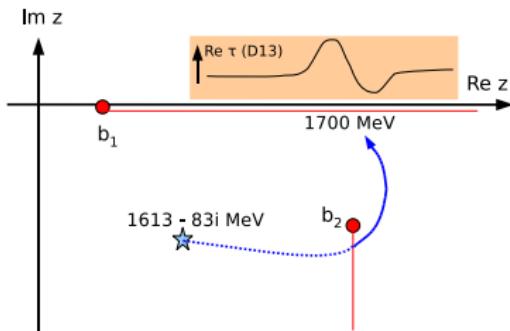
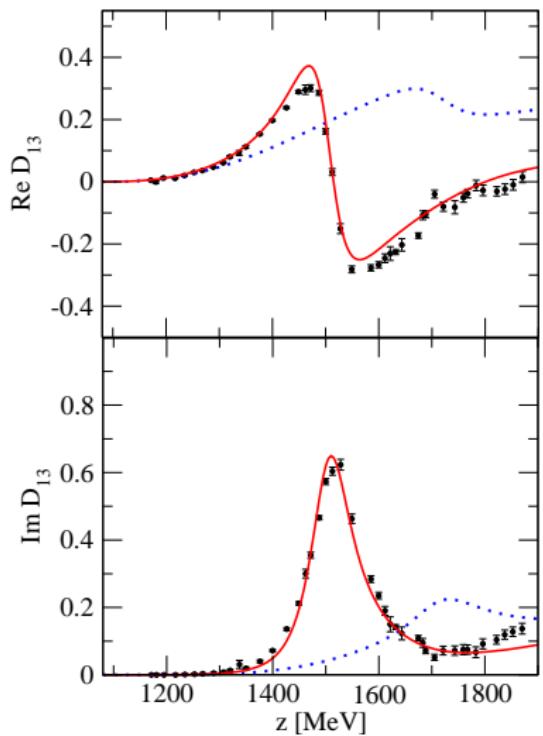
- ▶ Poles in T^{NP} may occur \Rightarrow pole repulsion in $T = T^{\text{NP}} + T^{\text{P}}$!



The D_{13} partial wave

◀ back

The $N^*(1520)$ and a dynamically generated pole in T^{NP} .



- ▶ T^{NP} : no s -channel $N^*(1520)$.
- ▶ Pole in T^{NP} on 3rd ρN sheet.
- ▶ On **physical** axis visible through branch point b_2 .
- ▶ Pole invisible in full solution.
→ We do not identify it with a dynamically generated $N^*(1700)$.

[Ramos, Oset, arXiv:0905.0973 [hep-ph]]

g_{fi} und h_{fi} in JLS-Basis:

$$\begin{aligned} g_{fi} &= \frac{1}{2\sqrt{k_f k_i}} \sum_j (2j+1) d^j_{\frac{1}{2} \frac{1}{2}}(\theta) \left[\tau^{j(j-\frac{1}{2})\frac{1}{2}} + \tau^{j(j+\frac{1}{2})\frac{1}{2}} \right] \cos \frac{\theta}{2} \\ &\quad + \frac{1}{2\sqrt{k_f k_i}} \sum_j (2j+1) d^j_{-\frac{1}{2} \frac{1}{2}}(\theta) \left[\tau^{j(j-\frac{1}{2})\frac{1}{2}} - \tau^{j(j+\frac{1}{2})\frac{1}{2}} \right] \sin \frac{\theta}{2} \end{aligned}$$

$$\begin{aligned} h_{fi} &= \frac{-i}{2\sqrt{k_f k_i}} \sum_j (2j+1) d^j_{\frac{1}{2} \frac{1}{2}}(\theta) \left[\tau^{j(j-\frac{1}{2})\frac{1}{2}} + \tau^{j(j+\frac{1}{2})\frac{1}{2}} \right] \sin \frac{\theta}{2} \\ &\quad + \frac{i}{2\sqrt{k_f k_i}} \sum_j (2j+1) d^j_{-\frac{1}{2} \frac{1}{2}}(\theta) \left[\tau^{j(j-\frac{1}{2})\frac{1}{2}} - \tau^{j(j+\frac{1}{2})\frac{1}{2}} \right] \cos \frac{\theta}{2} \end{aligned}$$

Observables

◀ back

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{k_f}{k_i} (|g_{fi}|^2 + |h_{fi}|^2) \\ &= \frac{1}{2k_i^2} \frac{1}{2} \cdot \left(\left| \sum_j (2j+1) (\tau^{j(j-\frac{1}{2})\frac{1}{2}} + \tau^{j(j+\frac{1}{2})\frac{1}{2}}) \cdot d_{\frac{1}{2}\frac{1}{2}}^j(\Theta) \right|^2 \right. \\ &\quad \left. + \left| \sum_j (2j+1) (\tau^{j(j-\frac{1}{2})\frac{1}{2}} - \tau^{j(j+\frac{1}{2})\frac{1}{2}}) \cdot d_{-\frac{1}{2}\frac{1}{2}}^j(\Theta) \right|^2 \right)\end{aligned}$$

$$\vec{P}_f = \frac{2Re(g_{fi}h_{fi}^*)}{|g_{fi}|^2 + |h_{fi}|^2} \cdot \hat{n}$$

$$\beta = \arctan \left(\frac{2Im(h_{fi}^* g_{fi})}{|g_{fi}|^2 - |h_{fi}|^2} \right)$$

